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Selection of scales in pattern-forming dynamics

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In the strongly nonlinear regime many pattern-forming systems, such as premixed flames, film flows, and three-dimensional hydrodynamical flows, exhibit a remarkable nonlinear phenomenon: the resulting patterns are substantially longer than the wavelength of the linearly most unstable mode. Usually such an inverse cascade, or coarsening is attributed to high-order nonlinear effects. We show, however, that the coarsening may be well described by a proposed weakly nonlinear evolution equation. The key of the model is the dispersion relation, which, being the kernel of Fourier convolution operator, captures the essential properties of strong instabilities in nonlinear systems.

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One of the most basic and interesting problems of pattern selection theory is to predict the length-scale of experimentally observed structures. Typically, the stability properties of the underlying physical system depend on a control parameter R; e.g., it might be the Reynolds or Rayleigh number in hydrodynamics. Let us say that at $R=R_c$ the system becomes unstable. When the system is only slightly unstable, $0<R-R_c\ll 1$, the length-scale of the formed patterns is close to the length of the most unstable (in linear description) mode. Such pattern dynamics is well described by one of the suitable well-known model equations [1,2]; for instance, Kuramoto-Sivashinsky (KS), Kawahara, and dissipation-modified KdV equations describe patterns induced by the long-wavelength instabilities. In the case of short-wavelength instabilities, the Ginzburg-Landau equation often appears.

When the instability becomes strong, $R-R_c\simeq O(1)$ or $R-R_c\gg 1$, the physical systems often exhibit a substantial stretching of the patterns compared with the length of the most unstable mode. In some contexts, such stretching is known as coarsening, or inverse energy cascade. The well-known examples of the coarsening are the (i) appearance of large cusplike waves in combustion [3], (ii) alpha-effect in three-dimensional hydrodynamics [4], and (iii) formation of long solitary waves in thin liquid films [5,6].

Traditionally, the phenomenon of coarsening is attributed to the impact of higher-order nonlinear effects. In conventional techniques of multiscale, or long-wavelength asymptotic expansions, the higher-order terms become important for instabilities far from the onset. However, the account of the higher-order nonlinearities usually allows one to extend the applicability of the conventional evolution equations only for R slightly exceeding R_c, and in most of cases is not applicable for the strong instabilities.

We propose an alternative explanation of the coarsening in nonlinear systems. The idea of proposed description consists of the following.

We want to extract common simple properties shared by various physical systems with coarsening for strong instabilities, with the hope that these properties will be responsible for the coarsening. With necessity, such approach should be *ad hoc*, since the conventional long-wavelength expansions do not work well for strong instabilities. Fortunately, careful investigation of a few nonlinear systems shows the direction of the search.

We start from the linear stability problem, where the variable u is represented as

$$u\sim e^{ikx+\omega t}, \tag{1}$$

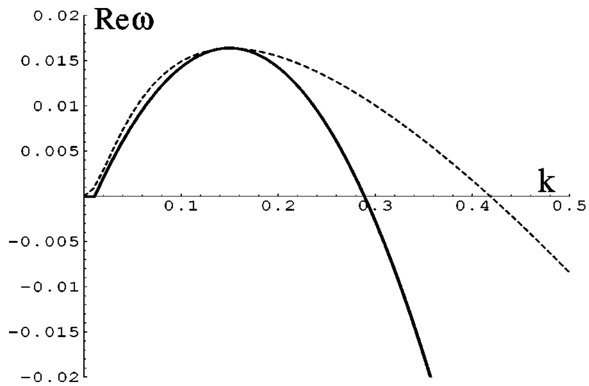

FIG. 1. Plot of Eq. (7) for $a=0.128, b=0.85, \varepsilon=0.022$ (solid line); exact dispersion relation for the downflowing film for conditions of [5] (dashed line).

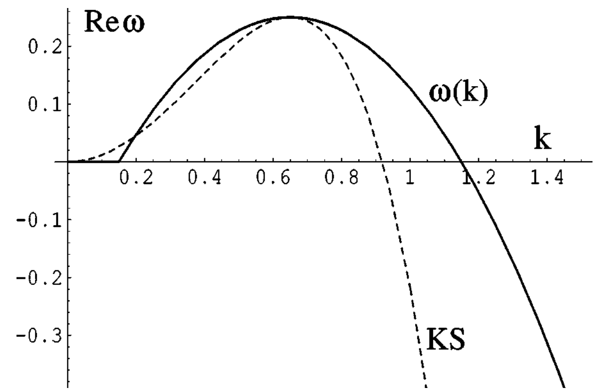

FIG. 2. Comparison of typical dispersion relation (6) for the KS equation (dashed line) with Eq. (7) for $\varepsilon=0.15$ (solid line).

with real wave number k and complex frequency ω. This form provides the first insight into the underlying dynamics. Dispersion relation $\omega(k)$ governs the dynamics of linear waves [7],

$$u_t+\int_{-\infty}^{\infty}-\omega\ e^{ik(x-y)}\ u(y,t)dy\ dk=0. \tag{2}$$

For weak instabilities, $\omega(k)$ may be typically well approximated by the power series in k. Technically, this may be done by multiscale asymptotic expansions [8], or by long-wavelength expansions [9]. Replacing ω by its approximation in terms of power series of k allows one to reduce the Fourier integral in Eq. (2) to a comfortable differential operator.

It is a very common and distinguishable property of the linear stability problem, that the dispersion relation $\omega(k)$ for strong instabilities dramatically differs from that for weak instabilities. The manifestation of the difference is that radius of convergence of power series expansion of $\omega(k)$ is typically small for instabilities far from the onset. As a result, $\omega(k)$ for strong instabilities *cannot* be well approximated by appropriate expansions in power series of k.

Therefore, the description of the linear stage of the relevant dynamics for strong instabilities should include the full Fourier integral (2), which is the basic classical idea for the description of nonlinear processes [7].

As for nonlinearity, we take the simplest kinematic term uu_x. This term appears in Burgers, KdV, KS, Kawahara equations, the description of weak turbulence [10], and other models [7]. In combustion [3,8] and thin films [2,11], this nonlinearity is sufficient to reproduce the main features of the pertinent dynamics.

Combining Eq. (2) with kinematic nonlinearity uu_x, we propose the following model equation:

$$u_t+uu_x+\int_{-\infty}^{\infty}-\omega\ e^{ik(x-y)}u(y,t)dy\ dk=0. \tag{3}$$

Now the problem of coarsening is reduced to the appropriate choice of $\omega(k)$. We want to find $\omega(k)$ such that (i) it is simple, (ii) includes some ‘‘control parameter’’ ε, which will govern the magnitude of the coarsening, and (iii) $\omega(k)$ should reasonably approximate the dispersion relations in the known equations with coarsening.

We will use equations for dynamics of premixed flames, and thin film to illustrate the choice of $\omega(k)$. The implicit formula defining $\omega(k)$ for the premixed flames is given in [8] [expression (17) on p. 1184]. We need only the understanding of its asymptotic behavior.

When the Lewis number is equal to unity and thermal expansion of combustion products is strong, the main term of the expression $\omega(k)$ reads [8]

$$\omega(k)\simeq|k|-k^2. \tag{4}$$

Dispersion relation (4) leads to the integral Sivashinsky equation with the full coarsening [3].

From the other side, below the critical Lewis number, the leading term of $\omega(k)$ for small thermal expansion is [8]

$$\omega(k)\simeq k^2-k^4. \tag{5}$$

The latter results in the KS equation, without coarsening.

Therefore, our dispersion relation in Eqs. (2) and (3) should be somehow close to Eqs. (4) and (5), when the control parameter ε changes.

It is remarkable that the thin film flows provide very similar criteria for the choice of $\omega(k)$.

When the Reynolds number R of the liquid film is slightly above the critical Reynolds number R_c, the dispersion relation has the following form [12,13]:

$$\omega\simeq(R-R_c)k^2-k^4, \tag{6}$$

which in turn results in the KS equation [2,14].

For the instability far from the onset, $\omega(k)$ could be found only numerically. Figure 1 shows the exact dispersion relation of downflowing liquid film for such a case [11] as a dashed line; here, Reynolds number $R=29$, Weber number $W=35$, and angle $\theta=6.4°$ (which are the experimental conditions in [5]).

It is remarkable that $\mathrm{Re}\,\omega(k)$ is very close to linear dependence $\omega(k)\simeq k$ for small k as in Eq. (4) (say, for $k<0.08$ on Fig. 1), though damping differs somewhat from k^2 in Eq. (4). Note that linear dependence, $\mathrm{Re}\,\omega\sim k$, arises in many hydrodynamical situations and typically leads to the emergence of large structures [3,4].

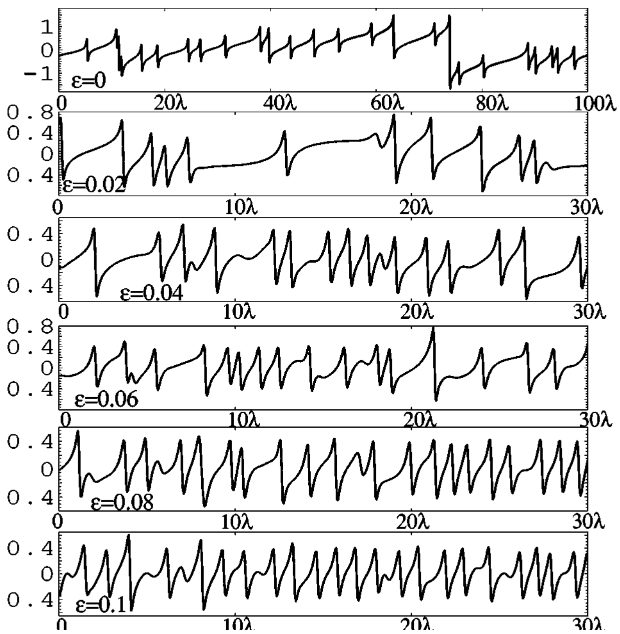

FIG. 3. Snapshots $u(x)$ in the developed regime for various values of ε.

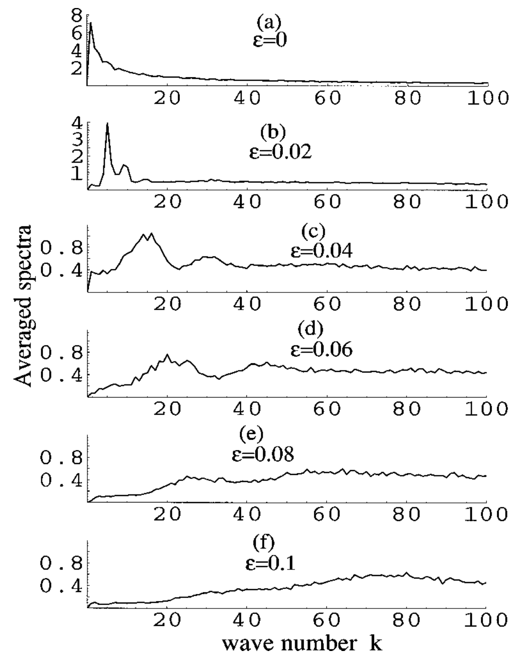

FIG. 4. Averaged spectra of $u(x)$ vs wave number k for various values of ε.

To mimic the outlined kind of linear behavior, we propose the following model dispersion relation:

$$\omega=\begin{cases} 0, & k<\varepsilon \\ b[a^2-(k-a-\varepsilon)^2], & k>\varepsilon, \end{cases} \tag{7}$$

with $\omega(-k)=\omega(k)$. Here ba^2 defines the height of the parabolic part of ω, a defines the width of the unstable part of ω, and ε defines the shift of parabolic part of ω from the origin. Function (7) is continuous, but its first derivative is discontinuous at $k=\varepsilon$. A comparison of the proposed $\omega(k)$ with exact dispersion relation for thin film is given in Fig. 1.

A comparison of Eq. (7) with KS dispersion relation (6) is shown in Fig. 2. Note that parameters a and b may be scaled out from Eq. (3) by a change of the coordinate and time scales, respectively; further, we take for simplicity b=1, $a=0.5$. *As a result, only a single parameter* ε *controls the spatiotemporal dynamics*.

The main idea of Eqs. (7) and (3) is that the parameter ε is assumed to control the length-scale of the emerging patterns. When $\varepsilon=0$ and therefore $\omega\sim|k|-k^2$, Eq. (3) may be reduced to the integral Sivashinsky equation [2,8] by the substitution $u=v_x$ and subsequent integration. Numerical simulation of the integral Sivashinsky equation shows a full coarsening [3]. For ε compared with a, relation (7) is close to the KS dispersion relation, Fig. 2. We assume therefore that in this case the dynamics produced by Eq. (3) will be close to that produced by the KS equation [15] without coarsening. As a result, we expect that variation of ε will mimic the transition from the full coarsening, to the degeneration of the coarsening.

The integral Sivashinsky equation admits exact solutions [16] and allows analytical investigation of the stability of these exact solutions [17]. Hopefully, Eq. (3) will allow-analytical solutions as well, or at least investigation by perturbations methods near the pole solutions of the integral Sivashinsky equation for small ε.

To check the above ideas, extensive numerical simulations of Eq. (3) were undertaken. We used periodic boundary conditions and standard pseudospectral technique. The Runge-Kutta fourth order scheme was used for the time advance with step 10^{-2}. The spatial discretization was such that the typical wavelength $\lambda=2\pi/k_c$ of the most unstable wave number k_c was covered by at least 20 points to ensure fair resolution of the computed solutions. Tests with smaller time steps and better resolution gave indistinguishable results. Random small-amplitude fields were used as initial conditions. The simulations were conducted on the long spatial interval of 100λ.

The results presented below use the following parameters: $\varepsilon=(0,2,4,6,8,10)\times10^{-2}$, $a=0.5$, $b=1$. The typical snapshots of $u(x,T)$ in the developed regime are shown in Fig. 3. Here $T=1000\tau_c$, where $\tau_c=1/\omega(k_c)$ is the typical time-scale of growth of the disturbances. The corresponding spectra, averaged over long time $2000\tau_c$, are shown in Fig. 4. As was mentioned above, for $\varepsilon=0$, Eq. (3) may be transformed

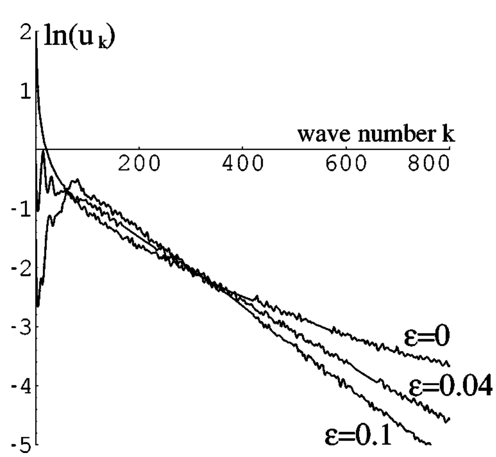

FIG. 5. Decay of spectra for various values of ε.

to the integral Sivashinsky equation. Numerical simulation shows [3] that in the developed regime the solutions of the integral Sivashinsky equation develop a large cusp structure with small irregular cusps. The size of the large cusp is equal to the whole computational domain. This kind of behavior may be clearly seen in Fig. 3(a), where the solution constitutes one sharp front, with smaller peaks around. Respectively, first Fourier harmonics of the solution is the leading one, Fig. 4(a). Therefore, the full coarsening occurs in this case.

For $\varepsilon=0.02$, the solution exhibits the typical intermittent structures, with the characteristic length substantially larger than λ, and *independent* of the length of computational domain (this was additionally checked on longer domains, and is not shown here). The corresponding spectrum has the maximum on the fifth Fourier component. Hence, compared to the pure linear dynamics, the *averaged* stretching in this case is $100/5=20$ times. For $\varepsilon=0$ the coarsening expands over the whole computational domain. In contrast to that, for $\varepsilon=0.02$ nonlinear *saturation* of the coarsening occurs. Note the diminishing of the amplitude of the leading Fourier mode, and distinctive appearance of the second harmonics compared with the case $\varepsilon=0$. For $\varepsilon=0.04$, the typical length-scale of the computed solutions shrinks compared the case $\varepsilon=0.02$, Figs. 3(b) and 3(c). Corresponding leading Fourier harmonics is now 18th; the averaged stretching is about $100/18\simeq 5$ times. Note that the second harmonics is relatively more prominent than for $\varepsilon=0.02$.

Further enlargement of ε leads to (i) a reversing of the role of the shorter and longer harmonics; namely, the shorter harmonics begins to be the leading one, and longer harmonics becomes to be a subharmonic [see Figs. 4(d)–4(f)]; (ii) a degeneration of stretching; and (iii) an appearance of KS-like patterns [see Figs. 3(e) and 3(f)].

Parameter ε also controls the rate of damping of Fourier spectrum for computed solutions; see Fig. 5. For larger ε, the short-wavelength modes are more damped.

We conclude that the dynamical response of Eq. (3) passes all stages of dynamical coarsening through variations of ε. This mimics the emergence and strengthening of the coarsening in real hydrodynamical flows when the control parameter becomes large enough.

The addition of KdV-like dispersion to Eqs. (7) and (3) entails regularization of the intermittent patterns in Fig. 3 (not shown here). A similar ordering influence of dispersion arises in the Kawahara equation [18].

As a result, Eq. (3) allows one to model the complicated process of coarsening appearing in many pattern-forming systems far from the onset of the instability, where conventional multiscale expansions are not very useful. We think that model (3) with dispersion relation (7) will have applications in a broad range of physical systems.

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